Topology in Physics 2018 - lecture 10

Marcel Vonk

April 17, 2018

We now begin to work our way towards index theorems: statements that relate the ***number of zero modes*** (in mathematics terminology: the dimension of the kernel) of certain differential operators on the one hand, to certain ***topological quantities*** on the other hand. As we will discuss later, the relations that these theorems give are very useful both in pure mathematics and in physics.

Before going into the statements of index theorems themselves, though, we need to define the type of operators that these theorems apply to. To this end, we first recall the notion of an adjoint operator, and then briefly discuss elliptic and Fredholm operators. We are then ready to state the Atiyah-Singer index theorem, which involves certain characteristic classes that we have not encountered yet - so we will discuss those in turn. Finally, we apply the Atiyah-Singer theorem to a simple example: the de Rham complex.

In what follows, we will quite closely follow chapter 12 (and parts of chapter 11) of Nakahara.

8.1 Adjoint operators

Let us begin with recalling the notion of an adjoint operator. Let V and W be two vector spaces (over $\mathbb{R}$ or $\mathbb{C}$) equipped with non-degenerate inner products $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$. Recall that the notion of "non-degenerate" means that

$$\text{if } \langle v_1, v_2 \rangle = 0 \qquad \text{for all } v_2, \tag{8.1}$$

then this implies that $v_1 = 0$: the only vector that has inner product 0 with any other vector is the zero-vector itself. A well-known corollary of this is that if one knows $\langle v_1, v_2 \rangle$ for any v_2, this *uniquely* determines the vector v_1. (To prove this, consider the difference of v_1 with another $\tilde{v}_1$ with the same property, and show that this difference must be the zero vector.)

Now, let us consider a linear operator $D : V \to W$. One defines its *adjoint operator* $D^\dagger : W \to V$ as follows: for a given $w \in W$, $D^\dagger w$ is the unique vector in V for which

$$\langle D^\dagger w, v \rangle_V = \langle w, Dv \rangle_W \qquad \text{for all } v \in V. \tag{8.2}$$

Since this equation describes the inner product of $D^\dagger w$ with any $v \in V$, by the corollary mentioned above, $D^\dagger w$ is indeed determined uniquely if it exists. To prove existence, note that the inner product of $D^\dagger w$ with an orthonormal basis for V determines an existing $D^\dagger w$, and that by linearity indeed the $D^\dagger w$ has the required inner product with *all* $v \in V$.

The canonical example of an adjoint operator to keep in mind is the case where V is $\mathbb{R}^m$, W is $\mathbb{R}^n$, and D is given by an $n \times m$ matrix. Using the standard inner product on V,

$$\langle v_1, v_2 \rangle_V = (v_1)^T \cdot v_2 \tag{8.3}$$

and similarly for W, we note that

$$\langle w, Dv \rangle_W = (w)^T \cdot (Dv) = (D^T w)^T \cdot v = \langle D^T w, v \rangle_V \tag{8.4}$$

for any $v \in V$, $w \in W$, and so in this case

$$D^\dagger = D^T \tag{8.5}$$

is simply the transposed matrix. This example may clear up some confusion that sometimes arises about the fact that a linear operator D from a "large" to a "small" space can uniquely determine an operator from the "small" to the "large" space, and that moreover this operator is well-defined even if the operator D is not invertible: note that in this example, a linear operator D from V to W is part of the nm-dimensional space $\mathrm{Mat}_{n\times m}(\mathbb{R})$, and that its adjoint is simply its transpose in the space $\mathrm{Mat}_{m\times n}(\mathbb{R})$, which is also an nm-dimensional space.

In what follows, we will be interested in linear (differential) operators on *infinte* dimensional vector spaces of functions or sections of bundles, but the concept of an adjoint operator applies to those cases just as well. Before discussing the infinite dimensional case further, let us notice an interesting property that exists already in the finite dimensional example above. Let us without loss of generality consider the case where $n > m$. Then a generic D can be "diagonalized" – that is, after a linear basis transformation it can be written as

$$D = \begin{pmatrix} \lambda_1 & & \emptyset \\ & \ddots & \\ \emptyset & & \lambda_m \\ & & \\ & \emptyset & \\ & & \end{pmatrix} \tag{8.6}$$

The bottom of this matrix consists of $n - m$ rows of zeroes, and therefore when all $\lambda_i \neq 0$, we see that $\dim\ker D = n - m$. By taking the adjoint (transpose), we moreover see that in this generic case, $\dim\ker D^\dagger = 0$. However, there are special loci in $\mathrm{Mat}_{n\times m}(\mathbb{R})$ where k of the λ_i become zero. This clearly increases the dimensions of *both* kernels by k, and so we see that whereas the individual kernels may vary in size, the statement that

$$\dim\ker D - \dim\ker D^\dagger = n - m \tag{8.7}$$

is always true.[1] The left hand side of the above equation is our first example of an *index*. We see that this index is much more "robust" under deformations of the operator D than the individual kernel dimensions of D and $D^\dagger$. Our aim is to generalize the left hand side of the above expression to (differential) operators on infinite-dimensional spaces, and the "robust" right-hand side to certain topological invariants.

8.2 Elliptic and Fredholm operators

Let us now consider two vector bundles, E and F, over the same manifold M, say with $\dim(M) = m$. We are interested in the case where our linear operator D is a differential operator that maps sections of E to sections of F:

$$D : \Gamma(M, E) \to \Gamma(M, F). \tag{8.8}$$

By "differential operator", we simply mean that if D is written out in coordinates, it is a sum of derivative operators with position-dependent coefficients. It is useful to write this out in detail. Let $U \subset M$ be a topologically trivial patch of M, and introduce coordinates x^μ, $1 \leq \mu \leq m$ on U. Since E and F are trivial over U, we can also find a basis of sections (that is, a frame) $\{e_a(x)\}$ for E and $\{\hat{e}_\alpha(x)\}$ for F. Note that we do not require the fiber dimensions to be the same, so we have $1 \leq a \leq k$ and $1 \leq \alpha \leq \ell$, with potentially different fiber dimensions k and ℓ.

Now, we can write out what we mean by the statement that D is a differential operator: for any $s(x) \in \Gamma(U, E)$, $Ds \in \Gamma(U, F)$ can be written in coordinates as

$$(Ds)^\alpha(x) = \sum_{|M| \leq N} \left(A^{(M)}\right)^\alpha{}_a(x) \partial_M s^a(x) \tag{8.9}$$

for a fixed integer N called the *order* of the operator. Of course, we assume that N is chosen minimally, so that the $\left(A^{(M)}\right)^\alpha{}_a(x)$ are not all identically zero for $|M| = N$. In the above expression, we use a multi-index notation where

$$M = (M_1, M_2, \ldots, M_m), \qquad M_i \in \mathbb{Z}_{\geq 0}, \tag{8.10}$$

and we write $|M| = \sum M_i$ and

$$\partial_M = \frac{\partial^{|M|}}{(\partial x^1)^{M_1} \cdots (\partial x^m)^{M_m}}. \tag{8.11}$$

In what follows, we are mostly interested in operators that are *elliptic*. Ellipticity is a condition on the highest derivative terms in the operator D. Let us consider the matrix

$$\Sigma(D, \xi)^\alpha{}_a(x) = \sum_{|M| = N} \left(A^{(M)}\right)^\alpha{}_a(x) \xi_M \tag{8.12}$$

[1]Strictly speaking, we did not prove the statement here for matrices that cannot be "diagonalized", but it can be shown to hold even in those cases

Here, $\xi = (\xi_1, \ldots, \xi_m)$ is a vector of auxiliary variables, and we use similar multi-index notation where

$$\xi_M = \xi_1^{M_1} \cdots \xi_m^{M_m}. \tag{8.13}$$

We can now define the notion of an ***elliptic operator***: D is called elliptic[2] if $\Sigma(D,\xi)^\alpha{}_a(x)$ is an invertible matrix for every $x \in M$ and for every (real) vector $(\xi_1, \ldots, \xi_m) \neq 0$. Of course, this implies in particular that a and α run over the same number of indices: we must have that the dimensions of the fibers of the bundles are equal, $k = \ell$.

The standard example of an elliptic operator is the Laplacian. Let us look at the Laplacian for functions on $\mathbb{R}^m$ for simplicity:

$$\Delta = \sum_{i=1}^m \left(\frac{\partial}{\partial x^i} \right)^2. \tag{8.14}$$

We are looking at functions here – that is, sections of the trivial, one-dimensional line bundle over $\mathbb{R}^m$ – so the symbol is now a 1×1 matrix:

$$\Sigma(\Delta, \xi)(x) = \sum_{i=1}^m (\xi_i)^2. \tag{8.15}$$

This "matrix" is invertible if the right hand side is nonzero, and this is clearly the case if some of the real entries ξ_i are nonzero. Therefore, the Laplacian is indeed an elliptic operator.

We need one more condition on our operators: we want them to be elliptical and ***Fredholm***. The generic definition of a Fredholm operator is slightly more involved, but in the case of the elliptic differential operators that we are interested in, we can characterize Fredholm operators in a very simple way: our operators are Fredholm if they have a *finite-dimensional* kernel and cokernel. Recall that the *cokernel* of an operator is defined as follows:

$$\text{coker } D \equiv \Gamma(M, F)/\text{Im}D. \tag{8.16}$$

Going back to our simple example of two finite-dimensional real vector spaces, it is clear that in that example

$$\text{coker } D \cong \ker D^\dagger \tag{8.17}$$

as both spaces can be represented by the subspace of the "target space" $W = \mathbb{R}^n$ that is perpendicular to ImD. In fact, as we will see in exercise 10.1, the above statement actually holds for *any* elliptic Fredholm operator and its adjoint.

The condition of being Fredholm is precisely what is needed to define the ***analytical index*** of an elliptic operator:

$$\text{ind}(D) = \dim\ker D - \dim\text{coker } D \tag{8.18}$$

[2] If you wonder about the reason for the terminology "elliptic": see exercise 10.2.

Of course, we can now also write this definition more symetrically using the adjoint operator:

$$\mathrm{ind}(D) = \dim\ker D - \dim\ker D^\dagger \tag{8.19}$$

Just like in our simple finite-dimensional example, this analytical index turns out to be very "robust" against deformations of the operator or the bundles. Our goal in this lecture is to make this robustness precise. For this, we first need to slightly extend our notion of an index from a single operator to operators acting on a complex.

8.3 The index of a complex

In what we have seen so far – for example when we introduced the exterior derivative – operators usually act within a *complex*:

$$\cdots \to \Gamma(M, E_{i-1}) \xrightarrow{D_{i-1}} \Gamma(M, E_i) \xrightarrow{D_i} \Gamma(M, E_{i+1}) \to \cdots \tag{8.20}$$

where the canonical example is of course the case where $\Gamma(M, E_i) = \Omega^i(M)$ are the i-forms and $D_i = d$ is the exterior derivative. We will dow be interested in the case where all operators in the above complex are elliptic and Fredholm[3], and where of course we still have the property that

$$D_i D_{i-1} = 0. \tag{8.21}$$

Just like in the de Rham exterior derivative example, one can now generically define the cohomology groups

$$H^i(E, D) \equiv \ker D_i/\mathrm{im}\ D_{i-1}. \tag{8.22}$$

We now define the analytical index of our complex to be

$$\mathrm{ind}(D) = \sum_{i=1}^{m} (-1)^i \dim H^i(E, D). \tag{8.23}$$

How is this related to our previous definition of the analytical index for a single operator? To see this, let us introduce adjoint operators as before:

$$D_i^\dagger : \Gamma(M, E_{i+1}) \to \Gamma(M, E_i). \tag{8.24}$$

Using these operators, one can construct the "generalized Laplacians"

$$\Delta_i \equiv D_{i-1}D_{i-1}^\dagger + D_i^\dagger D_i. \tag{8.25}$$

We now use without proof the following theorem (the ***Hodge decomposition theorem***): every cohomology class in $H^i(E, D)$ has a unique "harmonic" representative ω for which $\Delta_i \omega = 0$. As a result of this theorem, we can rewrite the analytical index of a complex as

$$\mathrm{ind}(D) = \sum_{i=1}^{m} (-1)^i \dim\ker \Delta_i. \tag{8.26}$$

[3] As we will see later, this is actually *not* quite the case in the exterior derivative example, but it is the case in cohomology.

In the case where we only have a single operator D, we can construct a very simple complex as follows:

$$0 \xrightarrow{i} \Gamma(M, E) \xrightarrow{D} \Gamma(M, F) \xrightarrow{\pi} 0. \tag{8.27}$$

Since $ii^\dagger = \pi^\dagger \pi = 0$, the index of this complex according to our new definition is

$$\mathrm{ind}(D) = \dim \ker D^\dagger D - \dim \ker D D^\dagger. \tag{8.28}$$

However, note that if $\omega \in \ker D^\dagger D$, then $\langle \omega, D^\dagger D\omega \rangle = 0$ and therefore, using the definition of the adjoint, $\langle D\omega, D\omega \rangle = 0$. Thus, for a positive definite inner product, $D\omega = 0$ and we find that

$$\ker D^\dagger D = \ker D. \tag{8.29}$$

One similarly proves that $\ker DD^\dagger = \ker D^\dagger$, and therefore our new definition of the index of a complex reduces to the old definition in the case of a single operator:

$$\mathrm{ind}(D) = \dim \ker D - \dim \ker D^\dagger. \tag{8.30}$$

As a result, we can always use the definition for a complex, even if we are interested in a single operator only.

8.4 The Atiyah-Singer index theorem

Let us now formulate the Atiyah-Singer index theorem for the complex (E, D) over a manifold M (compact, without a boundary) that we have introduced in the previous section. The theorem states that the index can be computed as follows:

$$\mathrm{ind}(D) = (-1)^{m(m+1)/2} \int_M \mathrm{ch}\big(\oplus_r (-1)^r E_r \big) \frac{\mathrm{Td}(TM^{\mathbb{C}})}{e(TM)}. \tag{8.31}$$

The left hand side is the analytical index that we have defined above. On the right hand side, we see an integral over the manifold involving several characteristic classes in its integrand. We have not introduced all of these characteristic classes yet; they will be explained in some detail in the next subsection. For now, we only want to stress that the integrand on the right hand side is a differential form which generically has components of many different degrees. Implicitly hidden in the notation is the fact that of the integrand, one should pick out the part of top degree (that is, the m-form part, where $\dim M = m$) and integrate that part over the manifold, so that a well-defined and coordinate invariant result is obtained.

Before explaining the ingredients on the right hand side in more detail, however, we want to stress the fact that this theorem builds a bridge between to very different areas of mathematics. The index on the left hand side is an object that appears in ***analysis***: it tells us something about the properties of a differential operator. Such an operator may have "zero modes" (nontrivial elements that span its kernel), and a priori one would expect the space of zero modes (the kernel) to vary as one deformes either the operator or the bundles that it acts on. In fact, in general it does, but as soon as one compares the space of zero modes to the space of zero modes of the *adjoint* operator, the result becomes "robust", which we can now make very precise: the difference of the kernel dimensions is a ***topological invariant***! It only depends on the topology of the bundles, and not on any further details of the operator, not on a choice of metric, and so on. The Atiyah-Singer index theorem therefore relates analysis to the a priori very unrelated subject of topology.

Such a relation between different branches of mathematics is of course very interesting as a purely mathematical result, but what makes it even more interesting is that index theorems also have many interesting applications in physics. To get a feeling for this, let us name two:

1. In many applications in physics, we are interested in solutions to an equation of the form $Df = 0$. For example, such an equation may describe massless fields (think of the Klein-Gordon equation), solitons, instantons, specific solutions to the linearized Einstein's equations, and so on. Constructing solutions to such equations is often very hard, but it may already be interesting to know "how many" solutions there are – that is, what the dimension of the kernel of D is.
   Of course, we know now that this dimension may actually depend on the details of the operator, but the dimension of the kernel minus that of the cokernel is well-defined, and (assuming the former is the bigger one) we have seen that the situations where the cokernel is nonzero are actually special (higher codimension in parameter space), so that generically the index tells us what $d = \dim \ker D$ is.
   This can be used in many ways. For example, once one finds d different linearly independent zero modes, the generic problem is solved and one needs to look no further. Also, for many computations it suffices to know the number of zero mode solutions, and not their exact form – for example if one wants to know how energy will be distributed over the different modes, or if one wants to calculate an entropy. Finally, it may turn out that d is in fact a negative number, in which case we learn that generically there are *no* solutions to the problem, and that one can only hope to find solutions if the parameters of the problem are fine-tuned.

2. A special topic in physics where index theorems play an imprtant role is *supersymmetry*. In supersymmetry, say in quantum mechanics, there is an operator Q that maps bosonic states of the system to fermionic states and vice versa. It can be shown that on states with a *nonzero* mass, Q is in fact invertible, so that for every massive bosonic state there is precisely one fermionic state of the same mass. The same statement is not true for states with zero mass, however. What is true is that the massless bosonic states can be constructed as the zero modes of Q, and the massless fermionic states as zero modes of its adjoint $Q^\dagger$. Thus, the *difference* of the number of massles bosons and fermions is an index and does not change if the problem is deformed. The physical interpretation of this fact is that a pair consisting of a massive boson and its corresponding fermion can become massles upon deformation, thus increasing the dimension of both $\ker D$ and of $\ker D^\dagger$ by one. Vice versa, massless states can only

become massive if they "pair up" so that a new massive boson is paired to a new massive fermion.

8.5 More on characteristic classes

In lecture 7, we have encountered *characteristic classes*: cohomology classes on manifolds M that can be associated to vector bundles $E \to M$ in a topologically invariant way. We also saw how to construct those characteristic classes: for a fiber bundle with r-dimensional fiber, start from an invariant polynomial of degree k on $\mathrm{Mat}_r(\mathbb{C})$ and consider

$$P(F, \ldots, F) \in \Omega^{2k}(M) \tag{8.32}$$

where F is the curvature of a connection ∇ on E.

In the lectures, we have so far only encountered one type of characteristic classes: the *Chern classes* that can be defined as

$$\det\left(I + \frac{iF}{2\pi}\right) = c_0(F) + c_1(F) + \ldots \tag{8.33}$$

where $c_k(F)$ is the $2k$-form part one obtains from expanding the left hand side. The Chern classes are easiest to calculate if F is diagonal:

$$\frac{iF}{2\pi} = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_k \end{pmatrix} \tag{8.34}$$

where k is the dimension of the (complex) fiber, and the diagonal entries x_i are 2-forms. For such a diagonal F, the Chern classes are easily calculated from the observation that

$$\det\left(I + \frac{iF}{2\pi}\right) = \prod_{i=1}^{k}(1 + x_i) \tag{8.35}$$

and therefore

$$\begin{aligned} c_0 &= 1 \\ c_1 &= \sum_{i=1}^{k} x_i \\ c_2 &= \sum_{i<j} x_i x_j \\ &\vdots \end{aligned} \tag{8.36}$$

and so on.

Now, one may of course be worried that a generic F can not always be diagonalized, but for complex vector bundles the ***splitting principle*** helps us out. This principle (see lecture 7 for details) tells us that when it comes to computing characteristic classes, any complex vector bundle can be replaced by a sum of complex line bundles. After this replacement, F is clearly diagonal, with x_i being the first Chern class of the ith line bundle.

The splitting principle makes working with characteristic classes of complex vector bundles much easier. For example, recall the *Chern characters* $ch_n(F)$, a different type of characteristic class that was introduced in exercise 7.2. The Chern characters are defined as

$$ch(F) = \text{Tr } \left(e^{iF/2\pi}\right) = ch_0(F) + ch_1(F) + ch_2(F) + \ldots \tag{8.37}$$

where as for the Chern classes, $ch_n(F)$ is the form of degree $2n$ in the expansion of the left hand side. Using the splitting principle, the Chern characters are now very easily expressed in terms of the x_i, since for a diagonal F we have

$$ch(F) = \text{Tr } \left(e^{iF/2\pi}\right) = \sum_{i=1}^{k} e^{x_i}. \tag{8.38}$$

Using this expression we quickly find that

$$\begin{aligned} ch_0 &= k \\ ch_1 &= \sum_{i=1}^{k} x_i \\ ch_2 &= \frac{1}{2}\sum_{i=1}^{k} x_i^2 \\ &\vdots \end{aligned} \tag{8.39}$$

which in turn leads to relations such as

$$\begin{aligned} ch_1 &= c_1 \\ ch_2 &= \frac{1}{2}c_1^2 - c_2 \\ &\vdots \end{aligned} \tag{8.40}$$

and so on. Note that the Chern characters do not contain any "new" information as compared to the Chern classes: either class at degree $2n$ can be expressed in terms of the other classes at degrees $\leq 2n$. This is a generic statement, no matter which new characteristic classes one introduces. It comes from the fact that at degree $2n$ essentially one new invariant polynomial can be written down: $x_1^n + \ldots + x_k^n$, and any other invariant polynomial can be written as a sum of products of similar polynomials at lower degree. For this reason, it may not seem very useful to introduce new types of characteristic classes for complex vector bundles once one type (say the Chern classes) has been constructed. However, specific characteristic classes arise in specific cases, and it is often very useful to have the different "bookkeeping devices" at ones disposal. In fact, we already see this in the statement of the Atiyah-Singer index theorem, where the Chern *character* appears.

In the index theorem, we also see the *Todd class* appearing. Using the splitting principle, we can now very easily introduce this characteristic class: in terms of the x_i it equals

$$\mathrm{Td}(F) = \prod_{i=1}^{k} \frac{x_i}{1 - e^{-x_i}} = \mathrm{Td}_0(F) + \mathrm{Td}_1(F) + \mathrm{Td}_2(F) + \ldots \tag{8.41}$$

We leave it to the reader to write some $\mathrm{Td}_n(F)$ in terms of the x_i and in terms of, say, the Chern classes.

Finally, let us recall one characteristic class for *real* vector bundles that we have encountered in exercise 7.4: the *Euler class* of the tangent bundle. The curvature of a real vector bundle can not be diagonalized in general, but it can generally be written in block diagonal form as

$$\frac{iF}{2\pi} = \begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & 0 & x_2 & \\ & & -x_2 & 0 & \\ & & & & \ddots \end{pmatrix}. \tag{8.42}$$

In fact, one can use a trick and complexify the real bundle (replace its fibers $\mathbb{R}^k$ by $\mathbb{C}^k$), after which one can use the splitting principle to "diagonalize" the curvature and write

$$\frac{iF}{2\pi} = \begin{pmatrix} ix_1 & 0 & & & \\ 0 & -ix_1 & & & \\ & & ix_2 & 0 & \\ & & 0 & -ix_2 & \\ & & & & \ddots \end{pmatrix}. \tag{8.43}$$

The Euler class that was introduced in exercise 7.4 can now simply be written as[4]

$$e(TM) = \sqrt{\det\left(\frac{iF}{2\pi}\right)}. \tag{8.44}$$

When $k = \dim M$ is odd, the Euler class vanishes, but when k is even, we can now write it as

$$e(TM) = \prod_{i=1}^{\ell} x_i. \tag{8.45}$$

[4]Here, we kept using the notation F for the curvature, though of course the curvature 2-form of a tangent bundle is usually denoted by R. In fact, there is a slight mismatch in conventions when it comes to factors of i between "vector bundle language" and "Riemannian geometry language", so one would usually write $R = iF$.

where $\ell = k/2$ is half of the dimension of the manifold.

Now that we have (re-) introduced the Chern character, the Todd class and the Euler class, have we seen all of the ingredients that appear in the Atiyah-Singer index theorem. Ultimately, we want to give a "physics proof" of the theorem itself, but for now we want to start applying it in intersting examples. Let us finish this lecture by discussing one such example.

8.6 Example: the de Rham complex

Let us begin by repeating the Atiyah-Singer index theorem:

$$\mathrm{ind}(D) = (-1)^{m(m+1)/2} \int_M \mathrm{ch}\big(\oplus_r (-1)^r E_r \big) \frac{\mathrm{Td}(TM^{\mathbb{C}})}{e(TM)}. \tag{8.46}$$

We want to apply this to our favorite complex: the de Rham complex. A first question is of course: is the exterior derivative an elliptic Fredholm operator? It is not hard to see that it is *not* when we view it as an operator from $\Omega^p(M)$ to $\Omega^{p+1}(M)$ (for example, its kernel is not finite-dimensional), but we state without proof that the exterior derivative *is* in fact elliptic when acting on cohomology groups $H^p(M)$.

To be able to apply the index theorem, we need one slight adjustment from the usual de Rham complex: since the theorem applies to *complex* vector bundles, we need to complexify the bundles of which the p-forms are sections to $\wedge^p(T^*M)^{\mathbb{C}}$. This technicality will not lead to too much difficulty in what follows.

Recall that the left hand side of the index theorem for complexes is defined in terms of the dimensions of the cohomoloy groups[5]:

$$\mathrm{ind}(D) = \sum_{r=0}^{m} (-1)^r \dim H^r(M) \tag{8.47}$$

which is known as the *Euler characteristic* of M. The Euler characteristic is known to vanish in odd dimensions, so we are interested in the case where $m = 2\ell$, which is also the case in which as we saw $e(TM)$ is nonzero.

Now, let us compute the different ingredients on the right hand side of the index theorem. As before, we want to write all of these expressions in terms of "diagonalized" 2-forms x_i. The main computation to be done turns out to be the computation of the Chern character

$$\mathrm{ch}\big(\oplus_r (-1)^r \wedge^r (T^*M)^{\mathbb{C}} \big). \tag{8.48}$$

[5]It is not hard to show that the complexification does not change the dimensions of the cohomology groups.

As we have seen in ecercise 7.2, the Chern character is linear when taking direct sums of bundles, so this equals

$$\mathrm{ch}\big(\oplus_r (-1)^r \wedge^r (T^*M)^{\mathbb{C}}\big) = 1 - \mathrm{ch}((T^*M)^{\mathbb{C}}) + \mathrm{ch}(\wedge^2 (T^*M)^{\mathbb{C}}) - \ldots \tag{8.49}$$

Moreover, the Chern character is multiplicative for tensor products. For *wedge* products we obtain an extra factor of $1/r!$, but this can be removed by ordering the x_i:

$$\mathrm{ch}\big(\oplus_r (-1)^r \wedge^r (T^*M)^{\mathbb{C}}\big) = 1 - \sum_{i=1}^{m} e^{x_i} + \sum_{i<j} e^{x_i} e^{x_j} - \ldots \tag{8.50}$$

which can be shortened to

$$\mathrm{ch}\big(\oplus_r (-1)^r \wedge^r (T^*M)^{\mathbb{C}}\big) = \prod_{i=1}^{m} (1 - e^{x_i}). \tag{8.51}$$

We need to make one final adjustment to this resut. The above expression is written in terms of the x_i that diagonalize the curvature of the *cotangent* bundle T^*M. The other ingredients of the index theorem, however, are written in terms of the *tangent* bundle TM. In exercise 10.3, we will see that replacing the cotangent bundle by the tangent bundle simply replaces $x_i \to -x_i$, so if we want to write our result in terms of the x_i for the tangent bundle, we find

$$\mathrm{ch}\big(\oplus_r (-1)^r \wedge^r (T^*M)^{\mathbb{C}}\big) = \prod_{i=1}^{m} (1 - e^{-x_i}). \tag{8.52}$$

This is a very nice result, as this product appears precisely in the denominator of the Todd class:

$$\mathrm{Td}((TM)^{\mathbb{C}}) = \prod_{i=1}^{m} \frac{x_i}{1 - e^{-x_i}}. \tag{8.53}$$

Thus, we are left with a product of x_i only. This product needs to be divided by the Euler class of the tangent bundle, which is the square root of this product. That is, in the above product, every x_i appears twice – once with each sign – whereas in the Euler class, it appears only once. What is left after the division is precisely the Euler class again (the reader should check that all signs cancel), and so we can finally write the index theorem as

$$\mathrm{ind}(D) = \int_M e(TM). \tag{8.54}$$

Thus, in this example, the Atiyah-Singer index theorem gives us an integral representation of the Euler character $\chi(M)$ – a representation that we also encountered in exercise 7.4. In the next lecture, we will see further examples of results that can be obtained using the index theorem.